10. Permanent is VNP-Complete, Part 2

Sunday, September 24, 2023 5:31 AM

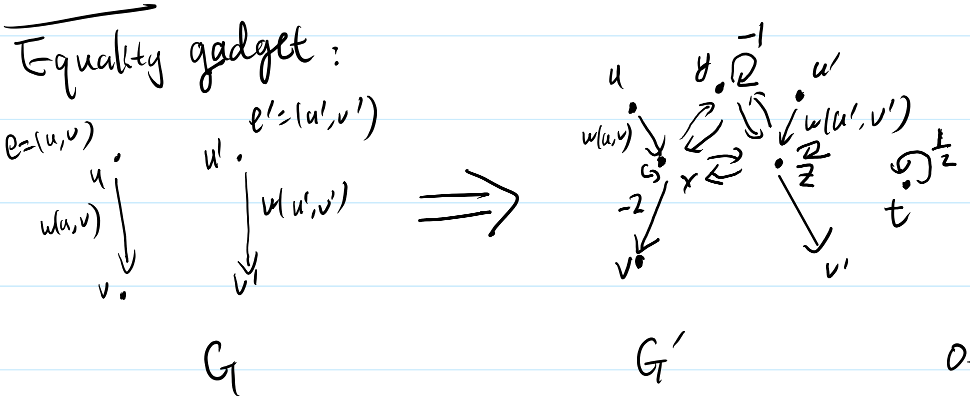
Last time: VNP = VNPe.

Today: Thm (Valiant): $\mathrm{PERM}_n = \mathrm{perm}((X_{ij}))$ is VNP-complete when $\mathrm{char}(\mathbb{F}) \neq 2$.

Note: If $\mathrm{char}(\mathbb{F}) = 2$, then PERM = DET since $-1 = 1$.

(However, there are VNP-complete polynomial families even when $\mathrm{char}(\mathbb{F}) = 2$.)

Equality gadget:



$w'(u,x) = w(u,v)$
$w'(u',z) = w(u',v')$
$w'(x,v) = -2$
$w'(y,y) = -1$
$w'(t,t) = \frac{1}{2}$ (requires $\mathrm{char}(\mathbb{F}) \neq 2$)
Other weights = 1.

Lemma 1: Let G be a weighted directed graph, $e, e' \in E(G)$.
Construct G' as above. Then

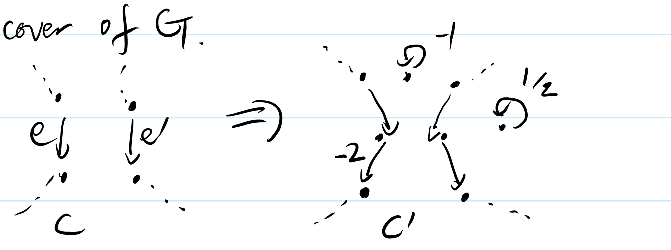
$$\mathrm{perm}(G') = \sum_{\substack{C \text{ cycle cover of } G \\ \text{either } e,e' \in C, \\ \text{or } e,e' \notin C.}} w(C)$$

we abuse the notation and use G' to denote the weighted adjacency matrix of G'.

Pf: Case by case analysis.

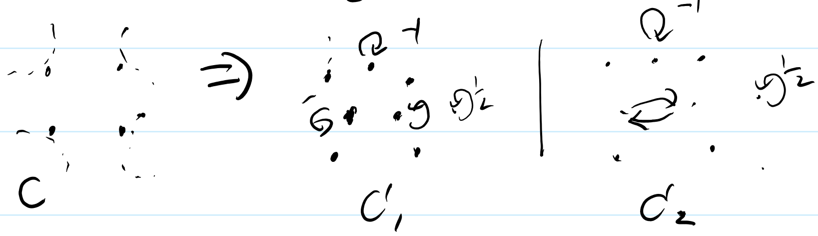
Let C be a cycle cover of G.

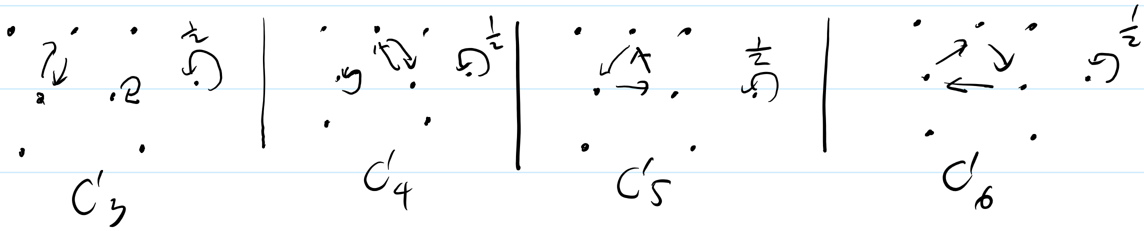
1) If $e, e' \in C$,



$w(C') = w(C)$.

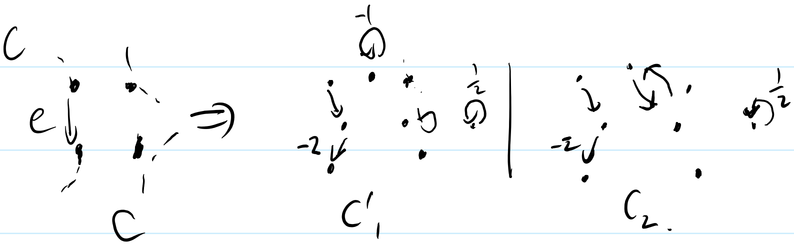
2) If $e, e' \notin C$,





$$\sum_{i=1}^{6} w(C'_i) = \frac{1}{2}(-1-1+1+1+1+1)\, w(C) = w(C).$$

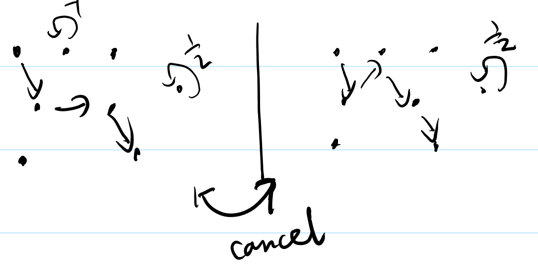
3) $e \in C$, $e' \notin C$.



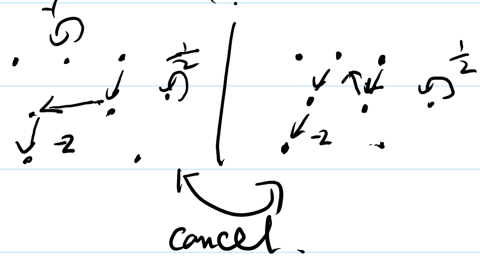
$$\sum_{i=1}^{2} w(C'_i) = \frac{1}{2}(-1+1)\, w(C) = 0.$$

4) $e \notin C$, $e' \in C$. Similar to (3), $\sum_{i=1}^{2} w(C'_i) = 0$.

Finally, G' may have cycle covers not corresponding to those of G.



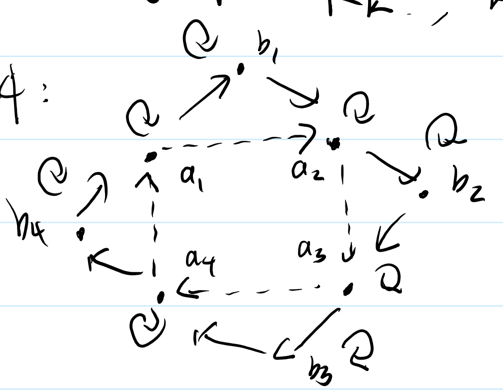
and similarly



Their weights cancel each other. $\square$

rosette graph R_k, $k > 0$.

k=4:



The inner edges $a_i \to a_{(i \bmod k)+1}$ are called "connector edges".

<u>Lemma 2</u>: Suppose S is a subset of connector edges of R_k.

1. If $S \neq \emptyset$, then $\exists$ exactly one cycle cover of R_k that contains S and no other connector edges.

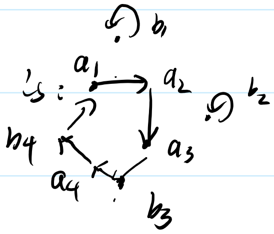
1. If $S \neq \phi$, then $\exists$ exactly one cycle cover of K_k that contains S and no other connector edges.

2. If $S = \phi$, then $\exists$ exactly two cycle covers of K_k that contain no connector edges.

<u>pf</u>: If $S \neq \phi$, when $(a_i \to a_{(i \bmod k)+1}) \notin S$, use the outer edges to connect a_i and $a_{(i \bmod k)+1}$

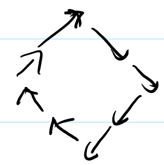
Then add self loops to b_i that are not covered.

E.g. for $k=4$, if $S = a_1 \to a_2 \to a_3$, then the cycle cover is:



It is easy to see it is the unique cycle cover.

When $S = \phi$, we have one cycle cover obtained this way:



But there is another cycle cover:

consisting of only self-loops.

It is easy to see these are the only two. $\square$

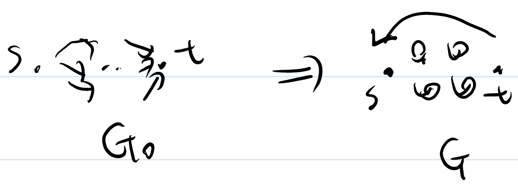
<u>Pf of Thm</u>: We already proved $\mathrm{PERM} \in \mathrm{VNP}$.

Let $(f_n) \in \mathrm{VNP}$. Then $f_n = \sum_{e=(e_1,\dots,e_{t(n)})} g_n(x_1,\dots,x_n,e_1,\dots,e_{t(n)})$ where $(g_n) \in \mathrm{VP}$ and t is p-bounded.

It suffices to show that $f_n \leq \mathrm{PERM}_m$ where $m \leq \mathrm{poly}(n)$.

As $\mathrm{VNP} = \mathrm{VNP}_e$ (from last time), we may assume $g_n \in \mathrm{VF}$.

So $g_n \in \mathrm{VBP}$, i.e., there is a poly(n)-size ABP G_0 computing g_n



The same proof that $g_n \leq_p \det$ shows that $g_n = \mathrm{perm}(G)$ (Lecture 7)

where G has size poly(n),

and weights of $E(G)$ are $x_1,\dots,x_n$, $y_1,\dots,y_{t(n)}$, and weights of $E(G)$ are $X_1, \dots, X_n$, $Y_1, \dots, Y_{t(n)}$, and constants in $\mathbb{F}$.

$$f_n = \sum_{e=(e_1,\dots,e_{t(n)})\in\{0,1\}^{t(n)}} g_n(X_1,\dots,X_n, e_1,\dots,e_{t(n)})$$

$$= \sum_{e} \sum_{\substack{\text{cycle cover} \\ C \text{ of } G}} w(C)$$ ← Let $m_C := w(C)$, which is a monomial in X_i's and Y_i's (with a coefficient).

$$= \sum_{\substack{\text{cycle cover} \\ C \text{ of } G}} \sum_{e} m_C(X_1,\dots,X_n, e_1,\dots,e_{t(n)})$$

$$= \sum_{\substack{\text{cycle cover} \\ C \text{ of } G}} 2^{I(C)} m_C(X_1,\dots,X_n,1,\dots,1)$$ where $I(C) := \#\, Y_i$ not appearing in m_C.

this is b/c if Y_i appears then we must let $e_i = 1$ to get a nonzero value.

If Y_i does not appear, then we may let $e_i = 0$ or 1.

call them $e_{i,1}, \dots, e_{i,k_i}$.

Suppose k_i edges of G are labeled with Y_i for $i = 1, \dots, t(n)$.

Let $H_i = R_{k_i}$, where every edge has weight 1.

Let G' = disjoint union of G and $H_1, \dots, H_{t(n)}$, (let $H_i := \emptyset$ if $k_i = 0$)

where the weights Y_i are replaced by 1 for all i.

Finally, for each $i = 1, \dots, t(n)$, and $j = 1, \dots, k_i$, connect $e_{i,j}$ with the j-th connector edge (call it $e'_{i,j}$) of $H_i \cong R_{k_i}$ using the equality gadget. Call the resulting graph G''.

Then
$$\mathrm{perm}(G'') \overset{\text{Lem 1}}{=} \sum_{\substack{\text{cycle cover } C' \\ \text{of } G', \text{ where} \\ \text{either } e_{i,j}, e'_{i,j} \in C' \\ \text{or } e_{i,j}, e'_{i,j} \notin C' \\ \text{for all } i,j.}} w(C') = \sum_{\substack{\text{cycle cover } C \text{ of } G \\ C_i \text{ of } H_i \\ \text{where } (C, C_1, \dots, C_{t(n)}) \\ \text{are consistent,} \\ \text{i.e. either } e_{i,j} \in C,\ e'_{i,j} \in C_i, \\ \text{or } e_{i,j} \notin C,\ e'_{i,j} \notin C_i.}} \left(w'(C) \cdot w(C_1) \cdots w(C_{t(n)})\right)$$

($w'(C) = m_C(X_1,\dots,X_n,\vec{1})$ since we replace Y_i by 1)

for all i,j.   i.e. either $e_{ij} \in C$, $e'_{i,j} \in C_i$, or $e_{ij} \notin C$, $e'_{i,j} \notin C_i$.

$$= \sum_{\substack{\text{consistent} \\ (C, C_1, \dots, C_{t(n)})}} m_C(x_1, \dots, x_n, 1, \dots, 1) \quad (\text{b/c } w(C_i) = 1).$$

$$= \sum_{\substack{\text{cycle cover} \\ C \text{ of } G}} m_C(x_1, \dots, x_n, 1, \dots, 1) \cdot \prod_{i=1}^{t(n)} (\# C_i \text{ consistent with } C)$$

By Lemma 2, $(\# C_i \text{ consistent with } C) = \begin{cases} 1 & Y_i \text{ appears in } m_C, \\ 2 & \text{otherwise.} \end{cases}$

So $\operatorname{perm}(G'') = \sum_{\substack{\text{cycle cover} \\ C \text{ of } G}} 2^{I(C)} m_C(x_1, \dots, x_n, 1, \dots, 1) = f_n$. $\square$

<u>Conjecture</u> (permanent vs. determinant) (perm_n) is not a p-projection of $(\det_n)$.

or equivalently, $VBP \neq VNP$.

(Since PERM is VNP-complete, and DET is VBP-complete).

We know $\operatorname{perm}_n \le_p \det_m$ when m is exponentially large in n.

<u>Thm</u> (Mignon–Ressayre '04) If $\operatorname{perm}_n \le_p \det_m$, then $m \ge n^2/2$.